

REMARK ON THE STABILITY OF THE LOG-SOBOLEV  
INEQUALITY FOR THE GAUSSIAN MEASURE

FILOMENA FEO

*Dipartimento di Ingegneria, Università degli Studi di Napoli “Parthenope”  
Centro Direzionale Isola C4  
80100 Naples, Italy.*

MARIA ROSARIA POSTERARO

*Dipartimento di Matematica e Applicazioni “Renato Caccioppoli”  
Via Cintia - Complesso Monte S. Angelo  
80100 Naples, Italy.*

CYRIL ROBERTO

*Université Paris Ouest Nanterre la Défense  
MODAL’X EA 3454, 200 avenue de la République  
92000 Nanterre, France.*

ABSTRACT. In this note we bound the deficit in the logarithmic Sobolev Inequality and in the Talagrand transport-entropy Inequality for the Gaussian measure, in any dimension, by mean of a distance introduced by Bucur and Fragalà.

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*E-mail addresses:* `filomena.feo@uniparthenope.it`, `posterar@unina.it`,  
`croberto@math.cnrs.fr`.

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## 1. INTRODUCTION

The log-Sobolev inequality asserts that, in any dimension  $n$  and for any smooth enough function  $f: \mathbb{R}^n \rightarrow \mathbb{R}_+^* := (0, +\infty)$ , it holds

$$(1.1) \quad \text{Ent}_{\gamma_n}(f) \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n,$$

where  $\gamma_n(dx) = \varphi_n(x)dx := (2\pi)^{-\frac{n}{2}} \exp\{-\frac{|x|^2}{2}\}dx$ ,  $x \in \mathbb{R}^n$ , is the standard Gaussian measure with density  $\varphi_n$ ,  $|x| = \sqrt{\sum_{i=1}^n x_i^2}$  stands for the Euclidean norm of  $x = (x_1, \dots, x_n)$  (accordingly  $|\nabla f|$  is the Euclidean length of the gradient) and  $\text{Ent}_{\gamma_n}(f) := \int_{\mathbb{R}^n} f \log f d\gamma_n - \int_{\mathbb{R}^n} f d\gamma_n \log \int_{\mathbb{R}^n} f d\gamma_n$  is the entropy of  $f$  with respect to  $\gamma_n$ . The constant  $1/2$  is optimal. Moreover, equality holds in (1.1) if and only if  $f$  is the exponential of a linear function, *i.e.* there exist  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}$  such that  $f(x) = \exp\{a \cdot x + b\}$ ,  $x \in \mathbb{R}^n$ . For simplicity we may write  $\varphi$  and  $\gamma$  for  $\varphi_1$  and  $\gamma_1$ .

The log-Sobolev inequality above goes back to Stam [36] in the late fifties. Later Gross, in his seminal paper [24], rediscovered the inequality and proved its fundamental equivalence with the so-called hypercontractivity property, a notion used by Nelson [33] in quantum field theory. Since then the log-Sobolev inequality attracted a lot of attention with many developments, applications and connections with other fields, including Geometry, Analysis, Combinatorics, Probability Theory and Statistical Mechanics. We refer to the monographs [1, 2, 25, 39, 27, 31] for an introduction. Finally, we mention that equality cases, in (1.1), appear in the paper by Carlen [12].

Very recently there has been some interest in the study of the stability of the log-Sobolev inequality (1.1). Namely the question is: can one bound the difference between the right and left hand side of (1.1) in term of the distance (in a sense to be defined) between  $f$  and the set of optimal functions? In other words, can one bound from below the *deficit*

$$(1.2) \quad \delta_{LS}(f) := \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n - \text{Ent}_{\gamma_n}(f)$$

in some reasonable way? We refer to [26, 7, 16] for various results in this direction. What is however currently lacking in the aforementioned literature is a result stating that, in fact,  $\delta_{LS}(f) \geq d(f, \mathcal{O})$  where  $\mathcal{O} := \{e^{a \cdot x + b}, a \in \mathbb{R}^n, b \in \mathbb{R}\}$  is the set of functions achieving equality in (1.1) and where  $d$  is some distance.

Our aim is to give a result in this direction, using a distance introduced by Bucur and Fragalà in [11] that we recall now.

*Bucur and Fragalà's construction of a distance modulo translation.* In this section we recall the procedure of Bucur and Fragalà [11] to define a distance (modulo translation) in dimension  $n$  starting with a distance (modulo translation) in dimension 1. We first give the definition of a *distance modulo translation*.

Let  $\mathcal{S}_n$  be some set of non-negative functions defined on  $\mathbb{R}^n$ . A mapping  $m: \mathcal{S}_n \times \mathcal{S}_n \rightarrow \mathbb{R}_+$  is said to be a distance modulo translation (on  $\mathcal{S}_n$ ) if (i)  $m$  is

symmetric, (ii) it satisfies the triangular inequality and (iii)  $m(u, v) = 0$  iff there exists  $a \in \mathbb{R}^n$  such that  $v(x) = u(x + a)$  for all  $x \in \mathbb{R}^n$ .

Now, given a direction  $\xi \in \mathbb{S}^{n-1}$  (the unit sphere of  $\mathbb{R}^n$ ), let  $x = (x', t\xi)$  be the decomposition of any point  $x \in \mathbb{R}^n$  in the direct sum of the linear span of  $\xi$  and its orthogonal hyperplane  $H_\xi := \{y \in \mathbb{R}^n : \langle \xi, y \rangle = 0\}$  (here  $\langle \cdot, \cdot \rangle$  stands for the Euclidean scalar product). Then, for all integrable function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , define  $f_\xi: \mathbb{R} \rightarrow \mathbb{R}, t \mapsto f_\xi(t) := \int_{H_\xi} f(x', t\xi) d\mathcal{H}^{n-1}(x')$ , where  $\mathcal{H}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure on  $H_\xi$ . Given a distance  $m$  modulo translation on some set  $\mathcal{S}$  of non-negative real functions, set

$$\mathcal{S}_n := \{f \in L^1(\mathbb{R}^n, \mathbb{R}_+), xf(x) \in L^1(\mathbb{R}^n, \mathbb{R}^n), f_\xi \in \mathcal{S} \text{ for all } \xi \in \mathbb{S}^{n-1}\}$$

and, for  $f, g \in \mathcal{S}_n$ ,

$$m_n(f, g) := \sup_{\xi \in \mathbb{S}^{n-1}} m(f_\xi, g_\xi).$$

In [11, Corollary 2.3], it is proved that  $m_n$  is a distance modulo translation on  $\mathcal{S}_n$ .

Also, Bucur and Fragalà [11] introduce the following distance modulo translation that we may use in the next sections. Set

$$\mathcal{B} := \left\{ u: \mathbb{R} \rightarrow \mathbb{R}_+^* : \text{continuous and } \int_{\mathbb{R}} u(x) dx = 1 \right\}.$$

Given two probability measures  $\mu(dx) = u(x)dx$ , and  $\nu(dx) = v(x)dx$ ,  $u, v \in \mathcal{B}$ , set  $T = F_\nu^{-1} \circ F_\mu$ , where  $F_\mu(x) := \int_{-\infty}^x u(y)dy$  and  $F_\nu(x) := \int_{-\infty}^x v(y)dy$  are the distribution functions of  $\mu$  and  $\nu$  respectively (observe that, since  $u \in \mathcal{B}$ ,  $F_\mu^{-1}$  is well defined and so does  $T'$  (note that  $T$  is increasing)).  $T$  is the transport map that pushes forward  $\mu$  onto  $\nu$ , i.e. the mapping satisfying  $\int_{\mathbb{R}} h(T) d\mu = \int_{\mathbb{R}} h dv$  for all bounded continuous function  $h$ . The following is a distance modulo translation on  $\mathcal{B}$  (see [11, Proposition 3.5])

$$(1.3) \quad d(u, v) := \int \frac{|1 - T'|}{\max(1, T')} d\mu.$$

We denote by  $d_n$  and  $\mathcal{B}_n$  the distance modulo translation and the set of functions constructed by the above procedure, starting from  $d$  and  $\mathcal{B}$  in dimension 1.

## 2. STABILITY OF THE LOG-SOBOLEV INEQUALITY

In order to state our main result, we need first to give a precise statement for (1.1) to hold.

It is well-known that (1.1) holds for any  $f$  such that  $\int_{\mathbb{R}^n} |f| d\gamma_n + \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n < \infty$ , i.e.  $|f|^{1/2} \in H^1(\gamma_n)$ , see e.g. [10, Chapter 1]. By a density argument one can restrict (1.1), without loss, to all  $f$  positive (since  $|\nabla|f|| = |\nabla f|$  almost everywhere), and by homogeneity, we can assume furthermore that  $\int_{\mathbb{R}^n} f d\gamma_n = 1$ . We call  $\mathcal{A}_n$  the set of  $C^1$  functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}_+^*$  such that  $\int_{\mathbb{R}^n} f d\gamma_n = 1$  and  $\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n < \infty$ . It is dense in the set of all functions satisfying the log-Sobolev Inequality (1.1) and is contained in  $\mathcal{B}_n$ . We observe that the set of extremal functions (with the proper normalization)  $\{\exp\{a \cdot x - \frac{|a|^2}{2}\}, a \in \mathbb{R}^n\}$  is contained in  $\mathcal{A}_n$ .

We are now in position to state our main theorem (recall the definition of  $d_n$  from the previous section).

**Theorem 2.1.** *For all  $n$  and all  $f \in \mathcal{A}_n$  it holds*

$$\delta_{LS}(f) \geq \frac{1}{2} d_n(f\varphi_n, \varphi_n)^2.$$

Before moving to the proof of Theorem 2.1 which is very short and elementary, let us comment on the above result.

First, from the above result, we (partially) recover the cases of equality in the log-Sobolev inequality for the Gaussian measure [12]. Indeed,  $f \in \mathcal{A}_n$  achieves the equality in the log-Sobolev inequality iff  $\delta_{LS}(f) = 0$  iff  $f\varphi_n$  is a translation of  $\varphi_n$  iff  $f(x) = \exp\{-a \cdot x - \frac{|a|^2}{2}\}$  for some  $a \in \mathbb{R}^n$ . This is only partial since Theorem 2.1 do not deal with all functions satisfying the log-Sobolev inequality but only with  $f \in \mathcal{A}_n$ . There is in fact some technical issues here: the distance  $d$  is no more a distance modulo translation if  $F_\mu^{-1}$  ( $\mu$  is the one dimensional probability measure with density  $f$ ) is not absolutely continuous (a property that is guaranteed by the fact that, in the definition of  $\mathcal{A}_n$ , we impose the positivity of the functions), see [11, Remark 3.6 (i)]. Hence, a result involving the distance  $d_n$  cannot recover, by essence, the full generality of Carlen's equality cases [12]. However,  $\mathcal{A}_n$  is very close to cover the set of all functions satisfying the log-Sobolev inequality (in particular it is dense in such a space) and, to the best of our knowledge, there is no result in the current literature that gives a lower bound of the deficit involving a distance without any second moment condition.

The assumption  $f$  of class  $C^1$ , in the definition of  $\mathcal{A}_n$  can certainly be relaxed. Indeed, one only needs, in dimension 1, that  $F_\mu^{-1}$  is an absolutely continuous function [11, Proposition 3.5] (for  $d\mu(x) = f(x)\varphi(x)dx$ ,  $x \in \mathbb{R}$ ). In dimension  $n$ , such a property should hold for all directions  $\xi \in \mathbb{S}^{n-1}$ . For this reason, and as mentioned above, there is no hope to obtain the whole family of functions satisfying the log-Sobolev Inequality. Hence, we opted for an easy and clean presentation rather than for a more technical one (a weaker assumption on  $f$  would have led us to technical approximations in many places, that, to our opinion, play no essential role).

We also observe that our result does not capture the product character of the log-Sobolev inequality. Indeed, if one considers, on  $\mathbb{R}^n$ , a function of the form  $f(x) = h(x_1)h(x_2)\dots h(x_n)$ ,  $x = (x_1, \dots, x_n)$ , with  $h: \mathbb{R} \rightarrow \mathbb{R}_+$ , then it is not difficult to see that  $\delta_{LS}(f)$  is of order  $n$  (i.e.  $\delta_{LS}(f) = n\delta_{LS}(h)$ ) thanks to the tensorisation property of (1.1) (see e.g. [1, Chapter 1]), while  $d_n(f\varphi_n, \varphi_n)$  is of order 1 (i.e.  $d_n(f\varphi_n, \varphi_n) = d(h\varphi, \varphi)$ ). This mainly comes from our use of Bucur and Fragalà's quantitative Prekopa-Leindler Inequality which is also, by construction, 1 dimensional. See below for some results based on the tensorisation property of the log-Sobolev inequality.

*Proof of Theorem 2.1.* The proof is based on the approach of Bobkov and Ledoux [9] to the log-Sobolev inequality by mean of the Prékopa-Leindler Inequality, together with an improved version of the Prékopa-Leindler Inequality of Bucur and Fragalà [11]. Our starting point is the following result (see [11, Proposition 3.5]):

given a triple  $u, v, w: \mathbb{R}^n \rightarrow \mathbb{R}_+$  with  $u, v \in \mathcal{B}_n$  and  $\lambda \in [0, 1]$  that satisfy  $w(\lambda x + (1 - \lambda)y) \geq u(x)^\lambda v(y)^{1-\lambda}$  for all  $x, y \in \mathbb{R}^n$ , it holds

$$(2.1) \quad \int_{\mathbb{R}^n} w(x) dx - 1 \geq \frac{1}{2} \lambda^{1+\lambda} (1 - \lambda)^{2-\lambda} d_n(u, v)^2.$$

We stress that the constant  $\lambda$  in the right hand side of the latter is not given explicitly in [11], but the reader can easily recover such a bound following carefully the proof of [11, Proposition 3.5]. (Inequality (2.1) goes back to the seventies [29, 35] and has numerous applications in convex geometry and functional analysis. We refer to the monographs [6, 17, 39] for an introduction. We further mention that equality cases are given in [15], and refer to Ball and Böröczky [3, 4] for related results on the stability of the Prékopa-Leindler Inequality.)

Our aim is to apply (2.1) to a proper choice of triple  $u, v, w$ . Following [9], let  $f = e^g$  with  $g$  sufficiently smooth with compact support and  $\int_{\mathbb{R}^n} f d\gamma_n = 1$ ,  $\lambda \in (0, 1)$ , and set

$$u_\lambda(x) = \frac{e^{\frac{g(x)}{1-\lambda}} \varphi_n(x)}{\int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}} d\gamma_n}, \quad v(y) = \varphi_n(y) \quad \text{and} \quad w_\lambda(z) = e^{g_\lambda(z)} \varphi_n(z)$$

with

$$g_\lambda(z) := \sup_{\substack{x, y: \\ (1-\lambda)x + \lambda y = z}} \left( g(x) - \frac{\lambda(1-\lambda)}{2} |x - y|^2 \right) - (1-\lambda) \log \int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}} d\gamma_n.$$

The function  $g_\lambda$  is the optimal function such that it holds  $w_\lambda((1-\lambda)x + \lambda y) \geq u_\lambda(x)^{1-\lambda} v(y)^\lambda$ . Set  $h_\lambda(z) := \sup_{\substack{x, y: \\ (1-\lambda)x + \lambda y = z}} \left( g(x) - \frac{\lambda(1-\lambda)}{2} |x - y|^2 \right)$ . Then, by (2.1) above, we get

$$\int_{\mathbb{R}^n} e^{h_\lambda} d\gamma_n \geq \left( \int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}} \right)^{1-\lambda} \left( 1 + \frac{1}{2} \lambda^{1+\lambda} (1 - \lambda)^{2-\lambda} d_n(u_\lambda, v)^2 \right)$$

The aim is to take the limit  $\lambda \rightarrow 0$ . We observe that (see [9] for details), as  $\lambda$  tends to zero

$$\left( \int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}} \right)^{1-\lambda} = \int_{\mathbb{R}^n} e^g d\gamma_n + \lambda \text{Ent}_{\gamma_n}(e^g) + o(\lambda)$$

and

$$\int_{\mathbb{R}^n} e^{h_\lambda} d\gamma_n = \int_{\mathbb{R}^n} e^g d\gamma_n + \frac{\lambda}{2(1-\lambda)} \int_{\mathbb{R}^n} |\nabla g|^2 e^g d\gamma_n + o(\lambda).$$

Therefore, dividing by  $\lambda$ , and taking the limit, we end up with

$$\liminf_{\lambda \rightarrow 0} \frac{1}{2} \lambda^{1+\lambda} (1 - \lambda)^{2-\lambda} d_n(u_\lambda, v)^2 + \text{Ent}_{\gamma_n}(e^g) \leq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla g|^2 e^g d\gamma_n.$$

We are left with the study of  $\liminf_{\lambda \rightarrow 0} d_n(u_\lambda, v)^2$ , since  $\lim_{\lambda \rightarrow 0} \lambda^{1+\lambda} (1 - \lambda)^{2-\lambda} = 1$ . For simplicity set  $u := u_0$  (i.e.  $u$  is the function  $u_\lambda$  defined above with  $\lambda = 0$ ). By the Lebesgue Theorem we observe that, for any direction  $\xi \in \mathbb{S}^{n-1}$ ,  $\lim_{\lambda \rightarrow 0} (u_\lambda)_\xi = u_\xi$ .

Hence, using again the Lebesgue Theorem (observe that, in the definition of  $d$ ,  $|1 - T'|/\max(1, T') \leq 1$ )

$$\liminf_{\lambda \rightarrow 0} d_n(u_\lambda, v) \geq \sup_{\xi \in \mathbb{S}^{n-1}} \liminf_{\lambda \rightarrow 0} d((u_\lambda)_\xi, v_\xi) = \sup_{\xi \in \mathbb{S}^{n-1}} d_n(u_\xi, (\varphi_n)_\xi) = d_n(e^\xi \varphi_n, \varphi_n).$$

The expected result follows for  $g$  sufficiently smooth. The result for a general  $f \in \mathcal{A}_n$  follows by an easy approximation argument (using again the monotone convergence Theorem and the Lebesgue Theorem), details are left to the reader.  $\square$

Next we derive from Theorem 2.1 a lower bound on the log-Sobolev inequality, in dimension  $n$ , that involves  $n$  times the one dimensional distance  $d$ . Such a result will capture on one hand the product structure of the inequality, but on the other hand the deficit will no more be bounded by a distance (modulo translation).

We need some notation. Given  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $i \in \{1, \dots, n\}$  and  $y_i \in \mathbb{R}$ , set  $\bar{x}^i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  and  $\bar{x}^i y_i := (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$  (so that  $\bar{x}^i x_i = x$ ). Then, for all functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and all  $x \in \mathbb{R}^n$ , we denote by  $f_{\bar{x}^i}: \mathbb{R} \rightarrow \mathbb{R}$  the one dimensional function defined by  $f_{\bar{x}^i}(y_i) := f(\bar{x}^i y_i)$ ,  $y_i \in \mathbb{R}$  (obviously  $f_{\bar{x}^i}(x_i) = f(x)$ ).

We may prove the following result.

**Corollary 2.2.** *For all  $n$  and all  $f \in \mathcal{A}_n$  it holds*

$$\delta_{LS}(f) \geq \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} d(f_{\bar{x}^i} \varphi, \varphi)^2 d\gamma_{n-1}(\bar{x}^i).$$

Now, by construction, if  $f(x) = h(x_1)h(x_2)\dots h(x_n)$ ,  $x = (x_1, \dots, x_n)$ , with  $h: \mathbb{R} \rightarrow \mathbb{R}_+^*$ , then both  $\delta_{LS}(f)$  and the right hand side of the latter are of (the correct) order  $n$ .

*Proof.* The proof uses the tensorisation property of the entropy. It is well known (see e.g. [1, Chapter 1]) that for any  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , it holds

$$\text{Ent}_{\gamma_n}(f) \leq \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} \text{Ent}_{\gamma}(f_{\bar{x}^i}) d\gamma_{n-1}(\bar{x}^i).$$

Hence, applying Theorem 2.1  $n$  times, we get (since  $f'_{\bar{x}^i}(x_i) = \frac{\partial f}{\partial x_i}(x)$ )

$$\begin{aligned} 2 \text{Ent}_{\gamma_n}(f) &\leq \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{f_{\bar{x}^i}^2(x_i)}{f_{\bar{x}^i}(x_i)} d\gamma(x_i) d\gamma_{n-1}(\bar{x}^i) - \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} d(f_{\bar{x}^i} \varphi, \varphi)^2 d\gamma_{n-1}(\bar{x}^i) \\ &= \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n - \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} d(f_{\bar{x}^i} \varphi, \varphi)^2 d\gamma_{n-1}(\bar{x}^i). \end{aligned}$$

The expected result follows.  $\square$

### 3. STABILITY OF THE TALAGRAND TRANSPORT-ENTROPY INEQUALITY

In this section we bound the deficit in the so-called Talagrand inequality, using again the distance  $d_n$  introduced by Bucur and Fragalà. Recall that (see e.g. [38]) the Kantorovich-Wasserstein distance  $W_2$  is defined as

$$W_2(\nu, \mu) := \inf_{\pi} \left( \iint |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}},$$

where the infimum runs over all couplings  $\pi$  on  $\mathbb{R}^n \times \mathbb{R}^n$  with first marginal  $\nu$  and second marginal  $\mu$  (i.e.  $\pi(\mathbb{R}^n, dy) = \mu(dy)$  and  $\pi(dx, \mathbb{R}^n) = \nu(dx)$ ). Talagrand, in his seminal paper [37], proved the following inequality: for all probability measure  $\nu$  on  $\mathbb{R}^n$ , absolutely continuous with respect to  $\gamma_n$ , it holds

$$(3.1) \quad W_2^2(\nu, \gamma_n) \leq 2H(\nu|\gamma_n),$$

where  $H(\nu|\gamma_n) := \int_{\mathbb{R}} \log \frac{d\nu}{d\gamma_n} d\nu$  if  $\nu \ll \gamma_n$  (whose density is denoted by  $d\nu/d\gamma_n$ ) and  $H(\nu|\gamma) = +\infty$  otherwise, is the relative entropy of  $\nu$  with respect to  $\gamma_n$ . Such an inequality, that is usually called Talagrand transport-entropy inequality, is related to Gaussian concentration in infinite dimension [37, 32, 19] (see the monographs [20, 28] for an introduction). It is known, since the celebrated work by Otto and Villani [34], that the log-Sobolev inequality (1.1) implies the Talagrand inequality (3.1) (in any dimension, see [8, 40, 5, 30, 21, 22, 23, 18] for alternative proofs and extensions).

The stability of (3.1) is also studied in [26, 16, 7, 14]. We may obtain that, as a direct consequence of the transport of mass approach of (3.1) by Cordero-Erausquin [13] and of the tensorisation property, one can bound from below, as for the log-Sobolev inequality, the deficit in (3.1) by the distance  $d_n$  defined by Bucur and Fragalà.

**Theorem 3.1.** *For all probability measure  $\nu$  on  $\mathbb{R}^n$  with continuous and positive density  $f \in \mathcal{B}_n$  with respect to the Gaussian measure  $\gamma_n$ , it holds*

$$(3.2) \quad \delta_{Tal}(f) := 2H(\nu|\gamma_n) - W_2^2(\nu, \gamma_n) \geq \frac{1}{2}d_n(f\varphi_n, \varphi_n)^2.$$

The above result together with theorem (2.1) somehow justify the use of the distance  $d_n$ . As for Theorem 2.1 the bound on the deficit is one dimensional and thus not of the correct order. This fact may become clear to the reader through the proof: we use some tensorisation property but apply a bound on the deficit only to one single coordinate.

*Proof.* The proof goes in two steps : we first prove the lower bound of the deficit in dimension 1, then we use a tensorisation procedure.

From [13] we can extract the following one dimensional inequality (here  $f: \mathbb{R} \rightarrow \mathbb{R}_+^*$ )

$$\delta_{Tal}(f) \geq \int_{\mathbb{R}} [T' - 1 - \log T'] d\gamma,$$

where  $T = F_\nu^{-1} \circ F_\gamma$  is the push forward of  $\gamma$  onto  $\nu$ . Using that  $s - 1 - \log s \geq \frac{1}{2} \left( \frac{1-s}{\max(1,s)} \right)^2$  and the Cauchy-Schwartz Inequality, we can conclude that

$$\delta_{Tal}(f) \geq \frac{1}{2} \int_{\mathbb{R}} \left( \frac{1-T'}{\max(1,T')} \right)^2 d\gamma \geq \frac{1}{2} \left( \int_{\mathbb{R}} \frac{|1-T'|}{\max(1,T')} d\gamma \right)^2 = \frac{1}{2} d(\varphi, f\varphi)^2,$$

which ends the proof of the first step (since  $d$  is symmetric).

Next, recall the tensorisation property of the Kantorovich-Wasserstein metric and of the relative entropy: for  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\nu(dx) = f(x)dx$ , we have

$$W_2^2(\nu, \gamma_n) \leq W_2^2(\nu_1, \gamma) + \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} W_2^2(\nu_{x_1, \dots, x_i}, \gamma) d\gamma_i(x_1, \dots, x_i)$$

and

$$H(\nu|\gamma_n) = H(\nu_1|\gamma) + \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} H(\nu_{x_1, \dots, x_i}|\gamma) d\gamma_i(x_1, \dots, x_i),$$

where we used the desintegration formula

$$\nu(dx_1, \dots, dx_n) = \nu_1(dx_1) \nu_{x_1}(dx_2) \nu_{x_1, x_2}(dx_3) \times \dots \times \nu_{x_1, \dots, x_{n-1}}(dx_n).$$

Now the supremum defining  $d_n(f\varphi_n, \varphi_n)$  is reached at some  $\xi \in \mathbb{S}^{n-1}$  that we may assume for simplicity and without loss of generality (since  $\gamma_n$  is invariant by rotation) to be the first unit vector of the canonical basis  $(1, 0, \dots, 0)$ . Using the tensorisation formulas above, applying the result we obtained in dimension 1, and (3.1)  $n-1$  times, we thus get

$$\begin{aligned} W_2^2(\nu, \gamma_n) &\leq W_2^2(\nu_1, \gamma) + \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} W_2^2(\nu_{x_1, \dots, x_i}, \gamma) d\gamma_i(x_1, \dots, x_i) \\ &\leq 2H(\nu_1|\gamma) - \frac{1}{2} d(f_1\varphi, \varphi)^2 + 2 \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} H(\nu_{x_1, \dots, x_i}|\gamma) d\gamma_i(x_1, \dots, x_i) \\ &= 2H(\nu|\gamma) - \frac{1}{2} d(f_1\varphi, \varphi)^2 = 2H(\nu|\gamma) - \frac{1}{2} d_n(f\varphi_n, \varphi_n)^2, \end{aligned}$$

where we set  $f_1$  for the density of  $\nu_1$  with respect to  $\gamma$ . By construction  $\nu_1$  is the first marginal of  $\nu$  so that  $f_1\varphi = (f\varphi_n)_\xi$ . This ends the proof.  $\square$

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